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RESEARCH

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A new model of economic order quantity associated with a generalized conformable differential-difference operator

Rabha W. Ibrahim^{1,2}, Rafida M. Elobaid^{3*}  and Suzan J. Obaiys⁴

*Correspondence:

relobaid2000@yahoo.com

³Deanship of Educational Services,
Department of General Sciences,
Prince Sultan University, Riyadh,
Saudi Arabia

Full list of author information is
available at the end of the article

Abstract

The economic order quantity (EOQ) is the order quantity that minimizes the total holding costs and ordering costs. In this effort, we propose a design for EOQ by employing a conformable differential-difference operator, which accepts to minimize the EOQ.

Keywords: Conformable fractional calculus; Fractional operator; Economic order quantity; Cost function

1 Introduction

The economic order quantity (EOQ) was introduced by Harris [1] and modified by Woolsey [2] and Selen and Wood [3]. Ibrahim and Hadid [4] involved the differential term in its formula. Jaber et al. [5] presented a structure based on a physical scheme called entropy. Many studies have indicated new mathematical modeling of EOQ (see [6–9]).

Ibrahim et al. [10] suggested a model for EOQ by using a fractal idea (local fractional calculus) with cost functions corresponding to time at a returning time. They used the formula of Tsallis fractal entropy. The study presented an optimization relaxation problem for scheming a limited interval covering the optimal series distance. In addition, they introduced developments to calculate the optimal measure and the optimal order phase. The estimated EOQ by using a fractal definition did not recognize the error in the EOQ model. Therefore, one can think about a fractional calculus with a controller.

Anderson and Ulness introduced a new type of conformable calculus (CC) [11] involving a control term. Moreover, Ibrahim and Jahangiri [12] extended CC to the complex plane to study the analytic solution of a class of complex differential equations. Based on CC, we generalize a differential-difference operator, type Dunkl of the first rank (CDD). Consequently, we employ this CDD to generalize the univex function, which we will utilize to minimize the EOQ system.

2 Preparing

In this section, we deal with the definitions of the important concepts of this effort.

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2.1 Conformable differential operator

In this place, we introduce the most recent definition of the conformable differential operator (CDO), which can be located in [11].

Definition 1 (Conformable differential operator) A differential operator \mathcal{D}^β , $\beta \in [0, 1]$ is conformable if and only if \mathcal{D}^0 is the identity operator and \mathcal{D}^1 is the ordinary differential operator. Particularly, the operator is conformable if and only if a differential function $\chi(t)$ satisfies

$$\mathcal{D}^0 \chi(t) = \chi(t) \quad \text{and} \quad \mathcal{D}^1 \chi(t) = \frac{d}{dt} \chi(t) = \chi'(t).$$

Moreover, in the theory of control systems, a proportional-differential controller for controlling resultant v at time t with two tuning criteria has the setting

$$v(t) = v_p \Sigma(t) + v_d \frac{d}{dt} \Sigma(t), \quad (1)$$

where v_p is the proportional gain, v_d is the derivative gain, and Σ is the error between the formal variable and the actual variable. Based on (1), Anderson and Ulness [11] presented the common idea of CDO.

Definition 2 (A special class of conformable differential operators) For two continuous functions $v_0, v_1 : [0, 1] \times \mathbb{R} \rightarrow (0, \infty)$, we attain

$$\mathcal{D}^\beta \chi(t) = v_1(\beta, t) \chi(t) + v_0(\beta, t) \chi'(t) \quad (2)$$

such that

$$\lim_{\beta \rightarrow 0} v_1(\beta, t) = 1, \quad \lim_{\beta \rightarrow 1} v_1(\beta, t) = 0, \quad v_1(\beta, t) \neq 0, \forall t, \beta \in (0, 1),$$

and

$$\lim_{\beta \rightarrow 0} v_0(\beta, t) = 0, \quad \lim_{\beta \rightarrow 1} v_0(\beta, t) = 1, \quad v_0(\beta, t) \neq 0, \forall t, \beta \in (0, 1).$$

Definition 3 The integral operator corresponding to \mathcal{D}^β is given by the following equality:

$$\int \mathcal{D}^\beta \chi(t) d_\beta t = \chi(t) + \wp e_0(t, t_0), \quad (3)$$

where $\wp \in \mathbb{R}$, $d_\beta t = \frac{dt}{v_0(t)}$, $v \neq 0$, and

$$e_0(t, \kappa) = \exp\left(-\int_\kappa^t \frac{v_1(\beta, \varsigma)}{v_0(\beta, \varsigma)} d\varsigma\right). \quad (4)$$

In our investigation, we request one of the following formulas of v_1 and v_0 :

$$v_1(\beta, t) = (1 - \beta)t^\beta, \quad v_0(\beta, t) = \beta t^{1-\beta}, \quad t \in (0, \infty), \quad (5)$$

$$\nu_1(\beta, t) = (1 - \beta)|t|^\beta, \quad \nu_0(\beta, t) = \beta|t|^{1-\beta}, \quad (6)$$

$$\nu_1(\beta, t) = \cos\left(\frac{\beta\pi}{2}\right)t^\beta, \quad \nu_0(\beta, t) = \sin\frac{\beta\pi}{2}t^{1-\beta}, \quad t \in (0, \infty), \quad (7)$$

or

$$\nu_1(\beta, t) = \cos\left(\frac{\beta\pi}{2}\right)|t|^\beta, \quad \nu_0(\beta, t) = \sin\frac{\beta\pi}{2}|t|^{1-\beta}, \quad t \in \mathbb{R} \setminus \{0\}. \quad (8)$$

Finally, the conformable inner product between two continuous functions χ and φ is given in the following formula:

$$\langle \chi, \varphi \rangle = \int_a^b \chi(t)\varphi(t)e_0(b, t) d_\beta t.$$

2.2 Differential-difference Dunkl operator

Dunkl operator is an assembly of differential-difference operators taking the formula of first rank (see [13, 14])

$$\Delta_\kappa \chi(t) = \chi'(t) + \kappa \left(\frac{\chi(t) - \chi(-t)}{t} \right), \quad \kappa \geq 0. \quad (9)$$

It extended a set of special functions and integral transforms in numerous variables linked with reflection groups. This type of operators has established many other operators. It is employed in the analysis of quantum body schemes. Lately, this operator is formulated in terms of fractional calculus and fractal (see [15, 16]). In this place, we aim to introduce the conformable Dunkl operator (CDD) by using (2). Therefore, we have the following operator:

$$\begin{aligned} \Delta_\kappa^\beta \chi(t) &= \mathcal{D}^\beta \chi(t) + \kappa \left(\frac{\chi(t) - \chi(-t)}{t} \right) \\ &= \nu_1(\beta, t)\chi(t) + \nu_0(\beta, t)\chi'(t) + \kappa \left(\frac{\chi(t) - \chi(-t)}{t} \right), \quad \kappa \geq 0. \end{aligned} \quad (10)$$

It is clear that when $\beta \rightarrow 1$, Eqs. (10) and (2) coincide. Moreover, the CDD has the following properties.

Proposition 2.1 *Let the CDD be given as in (10), where $\beta \in [0, 1]$. Then, for constants a , b , c and differential functions χ and φ , we have:*

- (i) $\Delta_\kappa^\beta [a\chi + b\varphi](t) = a\Delta_\kappa^\beta [\chi(t)] + b\Delta_\kappa^\beta [\varphi(t)];$
- (ii) $\Delta_\kappa^\beta c = c\nu_1(\beta, \cdot);$
- (iii) $\Delta_\kappa^\beta [\chi\varphi](t) = \chi(t)\Delta_\kappa^\beta \varphi(t) + \varphi(t)\Delta_\kappa^\beta \chi(t) - \chi(t)\varphi(t)\nu_1(\beta, t) + \kappa \left(\frac{\Phi(t) - \Phi(-t)}{t} \right),$ where

$$\Phi(t) := \chi\left(\frac{\varphi}{2} - 1\right) + \varphi\left(\frac{\chi}{2} - 1\right);$$

- (iv) $\Delta_\kappa^\beta \left[\frac{\chi}{\varphi} \right](t) = \frac{\varphi(t)\Delta_\kappa^\beta \chi(t) - \chi(t)\Delta_\kappa^\beta \varphi(t)}{\varphi^2(t)} + \frac{\chi(t)}{\varphi(t)}\nu_1(\beta, t) + K_\kappa(t),$ $\varphi \neq 0$, where

$$K_\kappa(t) := \kappa \left(\frac{\left[\frac{\chi}{\varphi} \right](t) - \left[\frac{\chi}{\varphi} \right](-t)}{t} \right) - \frac{\kappa}{\varphi^2(t)} \left(\frac{\chi(t) - \chi(-t)}{t} \right) - \frac{\kappa}{\varphi^2(t)} \left(\frac{\varphi(t) - \varphi(-t)}{t} \right).$$

Proof By using the definition of Δ_κ^β , we have

$$\begin{aligned}
 & \Delta_\kappa^\beta[a\chi + b\varphi](t) \\
 &= \mathcal{D}^\beta[a\chi + b\varphi](t) + \kappa \left(\frac{[a\chi + b\varphi](t) - [a\chi + b\varphi](-t)}{t} \right) \\
 &= v_1(\beta, t)[a\chi + b\varphi](t) + v_0(\beta, t)[a\chi + b\varphi]'(t) + \kappa \left(\frac{[a\chi + b\varphi](t) - [a\chi + b\varphi](-t)}{t} \right) \\
 &= a \left[v_1(\beta, t)\chi(t) + v_0(\beta, t)\chi'(t) + \kappa \left(\frac{\chi(t) - \chi(-t)}{t} \right) \right] \\
 &\quad + b \left[v_1(\beta, t)\varphi(t) + v_0(\beta, t)\varphi'(t) + \kappa \left(\frac{\varphi(t) - \varphi(-t)}{t} \right) \right] \\
 &= a\Delta_\kappa^\beta\chi(t) + b\Delta_\kappa^\beta\varphi(t).
 \end{aligned}$$

This completes the first part. For the second part, a calculation implies that

$$\Delta_\kappa^\beta c = \mathcal{D}^\beta c = v_1(\beta, t)c.$$

Now, we proceed to proving the multiplication rule under Δ_κ^β . We have

$$\begin{aligned}
 & \Delta_\kappa^\beta[\chi\varphi](t) \\
 &= v_1(\beta, t)[\chi\varphi](t) + v_0(\beta, t)[\chi\varphi' + \varphi\chi'](t) + \kappa \left(\frac{[\chi\varphi](t) - [\chi\varphi](-t)}{t} \right) \\
 &= \chi(t) \left(v_1(\beta, t)\varphi(t) + v_0(\beta, t)\varphi'(t) + \kappa \left(\frac{\varphi(t) - \varphi(-t)}{t} \right) \right) \\
 &\quad + \varphi(t) \left(v_1(\beta, t)\chi(t) + v_0(\beta, t)\chi'(t) + \kappa \left(\frac{\chi(t) - \chi(-t)}{t} \right) \right) - v_1(\beta, t)\chi(t)\varphi(t) \\
 &\quad + \kappa \left(\frac{[\chi(\frac{\varphi}{2} - 1) + \varphi(\frac{\chi}{2} - 1)](t) - [\chi(\frac{\varphi}{2} - 1) + \varphi(\frac{\chi}{2} - 1)](-t)}{t} \right) \\
 &:= \chi(t)\Delta_\kappa^\beta\varphi(t) + \varphi(t)\Delta_\kappa^\beta\chi(t) - v_1(\beta, t)\chi(t)\varphi(t) + \kappa \left(\frac{\Phi(t) - \Phi(-t)}{t} \right).
 \end{aligned}$$

Finally, we impose

$$\begin{aligned}
 & \Delta_\kappa^\beta \left[\frac{\chi}{\varphi} \right](t) \\
 &= v_1(\beta, t) \left[\frac{\chi}{\varphi} \right](t) + v_0(\beta, t) \left[\frac{\varphi\chi' - \chi\varphi'}{\varphi^2} \right](t) + \kappa \left(\frac{[\frac{\chi}{\varphi}](t) - [\frac{\chi}{\varphi}](-t)}{t} \right) \\
 &= \frac{v_1(\beta, t)[\varphi\chi](t) + v_0(\beta, t)[\varphi\chi' - \chi\varphi'](t)}{\varphi^2(t)} + \kappa \left(\frac{[\frac{\chi}{\varphi}](t) - [\frac{\chi}{\varphi}](-t)}{t} \right) \\
 &= \frac{\varphi(t)[v_1(\beta, t)\chi(t) + v_0(\beta, t)\chi'(t) + \kappa(\frac{\chi(t) - \chi(-t)}{t})]}{\varphi^2(t)} \\
 &\quad - \frac{\chi(t)[v_1(\beta, t)\varphi(t) + v_0(\beta, t)\varphi'(t) + \kappa(\frac{\varphi(t) - \varphi(-t)}{t})]}{\varphi^2(t)}
 \end{aligned}$$

$$\begin{aligned}
& + \kappa \left(\frac{[\frac{\chi}{\varphi}](t) - [\frac{\chi}{\varphi}](-t)}{t} \right) - \frac{\kappa}{\varphi^2(t)} \left(\frac{\chi(t) - \chi(-t)}{t} \right) - \frac{\kappa}{\varphi^2(t)} \left(\frac{\varphi(t) - \varphi(-t)}{t} \right) \\
& + v_1(\beta, t) \left(\frac{\chi}{\varphi} \right)(t) \\
& := \frac{\varphi(t) \Delta_{\kappa}^{\beta} \chi(t) - \chi(t) \Delta_{\kappa}^{\beta} \varphi(t)}{\varphi^2(t)} + \frac{\chi(t)}{\varphi(t)} v_1(\beta, t) + K_{\kappa}(t).
\end{aligned}$$

□

Proposition 2.2 *The integration by parts of Δ_{κ}^{β} satisfies the following integral:*

$$\begin{aligned}
\int_a^b \varphi(\varsigma) \Delta_{\kappa}^{\beta} \chi(\varsigma) e_0(b, \varsigma) d_{\beta} \varsigma &= \varphi(\varsigma) \chi(\varsigma) e_0(b, \varsigma) \Big|_a^b - \int_a^b \chi(\varsigma) [\mathcal{D}^{\beta} \varphi(\varsigma) \\
&\quad - v_1(\beta, \varsigma) \varphi(\varsigma)] e_0(b, \varsigma) d_{\beta} \varsigma + \kappa \langle \varphi, \Lambda \rangle,
\end{aligned} \tag{11}$$

where $\Lambda(\varsigma) = \frac{\chi(\varsigma) - \chi(-\varsigma)}{\varsigma}$.

Proof By the definition of Δ_{κ}^{β} , we have

$$\begin{aligned}
& \int_a^b \varphi(\varsigma) \Delta_{\kappa}^{\beta} \chi(\varsigma) e_0(b, \varsigma) d_{\beta} \varsigma \\
&= \int_a^b \varphi(\varsigma) \left[\mathcal{D}^{\beta} \chi(\varsigma) + \kappa \left(\frac{\chi(\varsigma) - \chi(-\varsigma)}{\varsigma} \right) \right] e_0(b, \varsigma) d_{\beta} \varsigma \\
&= \int_a^b \varphi(\varsigma) [\mathcal{D}^{\beta} \chi(\varsigma)] e_0(b, \varsigma) d_{\beta} \varsigma + \int_a^b \varphi(\varsigma) \kappa \left(\frac{\chi(\varsigma) - \chi(-\varsigma)}{\varsigma} \right) e_0(b, \varsigma) d_{\beta} \varsigma \\
&:= \int_a^b \varphi(\varsigma) [\mathcal{D}^{\beta} \chi(\varsigma)] e_0(b, \varsigma) d_{\beta} \varsigma + \kappa \int_a^b \varphi(\varsigma) \Lambda(\varsigma) e_0(b, \varsigma) d_{\beta} \varsigma.
\end{aligned}$$

By using Lemma 1.9 in [11] and the definition of the conformable inner product between two continuous functions, we have the desired conclusion. □

In this place, we note that the CDD is not commute operator, because the conformable operator \mathcal{D}^{β} is not commute operator [11] for a real case, while it is commute operator in a complex domain [12] for some classes of conformal function (has no zero derivative). Moreover, CDD is not invariant because the conformable operator \mathcal{D}^{β} is not invariant for a real case, while for the set of conformal functions in the open unit disk, the operator \mathcal{D}^{β} is invariant [12]. In the next subsection, we shall use the CDD to generalize the definition of the univex functions. These functions are useful in optimization analysis to minimize the EOQ.

2.3 Generalized univex function

A class of univex functions is a generalized class of convex functions [17]. This type of functions is usually used to solve multi-objective problems, especially in control theory. Multi-objective difficulties are reflected in control of space constrictions, aeronautical control scheme, and industrial development control, control of production, impulsive control problems and inventory, mechanics, economics, and mechanical engineering difficulties. Here, we aim to employ the CDD to introduce the conformable univex function (CUF).

Definition 4 For a real-valued function, χ is a univex function if and only if there are real-valued functions Ψ, f , and g satisfying the differential inequality

$$g(t, \tau) \Psi [\chi(t) - \chi(\tau)] \geq f(t, \tau) \chi'(t), \quad g \neq 0. \quad (12)$$

Now, by using the conformable operator \mathcal{D}^β defined in (3), we have the following conformable univex function (CUF).

Definition 5 For a real-valued function, χ is called a β -conformable univex function (CUF of order $\beta \in [0, 1]$) if and only if there are real-valued functions Ψ, f , and g satisfying the differential inequality

$$g(t, \tau) \Psi [\chi(t) - \chi(\tau)] \geq f(t, \tau) \mathcal{D}^\beta \chi(t), \quad g \neq 0. \quad (13)$$

Consequently, by utilizing the CDD (Δ_κ^β) , we have the generalized conformable univex function as follows.

Definition 6 For a real-valued function, χ is called a (β, κ) -conformable univex function of order $\beta \in [0, 1]$ and index $\kappa \geq 0$ if and only if there are real-valued functions Ψ, f , and g satisfying the differential inequality

$$g(t, \tau) \Psi [\chi(t) - \chi(\tau)] \geq f(t, \tau) \Delta_\kappa^\beta \chi(t), \quad g \neq 0. \quad (14)$$

2.4 Economic order quantity

The pattern of EOQ for the first time was produced by Ford W. Harris [1] as follows:

$$Q = \sqrt{\frac{2AB}{C}}, \quad (15)$$

where A indicates the annual demand quantity, B represents the fixed cost for each unit, and C takes the annual holding cost for each unit. In [4], the authors studied the marginal ordinary continuous case taking the form

$$Q'(t) = \Theta(Q(at), Q(bt), Q(ct)), \quad (16)$$

where a, b , and c are nonnegative EOQ model constants indicating to the regular variations in A, B , and C , while the fractal case of EOQ was studied in [10]. Here, we consider the CDD in Eq. (10) to generalize (16) as follows:

$$\begin{aligned} \Delta_\kappa^\beta Q(t) &= v_1(\beta, t) Q(t) + v_0(\beta, t) Q'(t) + \kappa \left(\frac{Q(t) - Q(-t)}{t} \right) \\ &= v_1(\beta, t) Q(t) + v_0(\beta, t) \Theta(Q(at), Q(bt), Q(ct)) + \kappa \left(\frac{Q(t) - Q(-t)}{t} \right). \end{aligned} \quad (17)$$

Because of a state in which the consumer uses an EOQ system to control the account of a failing item, when the shortage is not allowed (see [18]), we consider the following

modification:

$$\begin{aligned}\Delta_{\kappa}^{\beta} Q(t) &= v_1(\beta, t)Q(t) + v_0(\beta, t)\Theta(Q(at), Q(bt), Q(ct)) \\ &\quad + \kappa \left(\frac{Q(t) - Q(\tau)}{t - \tau} \right), \quad t > \tau.\end{aligned}\quad (18)$$

We need to estimate the optimal solution of units according to (17). That is, we minimize the total cost connected with the purchase, distribution, and storing of the manufactured goods. In the existence of a planned customer, who reacts optimally to discount program, the strategy of optimal quantity discount structure by the provider is complex and has to be done sensibly. This is principally happening when the request of the customer is itself unclear. An exciting result called the “reverse bullwhip” takes place where a rise in consumer request uncertainty really decreases order quantity indecision at the provider. There are different methods and theorems to minimize the total cost functions (minimize EOQ) such as the planning horizon theorem [19], stochastic processing [20], and fixed point theorem [4].

From the above information, one can reduce the EOQ model in (18) into the following optimization problem:

$$\begin{aligned}\text{Minimize } & \Delta_{\kappa}^{\beta} Q(t), \quad t \in J = [0, T] \\ \text{subject to } & Q_0 = q \leq 0,\end{aligned}\quad (19)$$

where q is the initial value of EOQ and Q is continuous for all $t \in J$. For this purpose, we need the following definitions.

Definition 7 For any solution $Q(t)$ for (19), a point $\eta \in \mathfrak{J} := \{\eta \in J : Q(\eta) \rightarrow 0\}$ is called an efficient point of (19) if there occurs no other point satisfying the inequality $Q(t) \leq Q(\eta)$. Furthermore, it is a weak efficient point if $Q(t) < Q(\eta)$, $\forall t \in J$.

Definition 8 A couple $(\beta, \kappa) - (\chi, \Upsilon)$ is called conformable univex of order β if and only if there are real-valued functions f_1, f_2, g_1 , and g_2 such that

$$g_1[\chi(t) - \chi(\tau)] \geq f_1(t, \tau) \Delta_{\kappa}^{\beta} \chi(t), \quad g_1 \neq 0,$$

and

$$-g_2[\Upsilon(t) - \Upsilon(\tau)] \geq f_2(t, \tau) \Delta_{\kappa}^{\beta} \Upsilon(t), \quad g_2 \neq 0.$$

Remark 1 The EOQ model accepts stable demand of a business, product, and immediate availability of items to be re-stocked. It does not account for regular or economic fluctuations. It accepts fixed costs of inventory units, ordering charges, and holding charges. Therefore, this term refers to the contraction in the economy. Consequently, the suggested model is the first model describing this issue by using Dunkl operator.

3 Results

In this section, we present some sufficient optimal hypotheses for a point to be an efficient solution of (19) under the conformable (β, κ) -univexity.

Theorem 3.1 *Let η be an initial point of the solution of the multi-objective problem (19) and ϵ_1 and ϵ_2 be two nonnegative constants achieving the following assumptions:*

- (A) $Q(\eta) = 0$ (the initial value of EOQ);
- (B) $\epsilon_1 f_1(t, \tau) \Delta_\kappa^\beta Q(t) + \epsilon_2 f_2(t, \tau) \Delta_\kappa^\beta q(t) > \epsilon_1 \|g_1\| + \epsilon_2 \|g_2\|$;
- (C) A couple $(\beta, \kappa) - (\chi, \Upsilon)$ is a conformable univex of order β ;

Then η is an efficient solution to minimize (19).

Proof Assume that η is not an efficient solution of (19). Therefore, there occurs $x \in \mathfrak{J}$ such that $Q(x) \leq Q(\eta)$. By utilizing assumption (C), we obtain

$$\epsilon_1 (f_1(x, \eta) \Delta_\kappa^\beta Q(x)) < \epsilon_1 \|g_1\| \quad (20)$$

and

$$\epsilon_2 (f_2(x, \xi) \Delta_\kappa^\beta q(x)) \leq \epsilon_2 \|g_2\|. \quad (21)$$

By taking the summation of (20) and (21), we have

$$\epsilon_1 (f_1(x, \eta) \Delta_\kappa^\beta Q(x)) + \epsilon_2 (f_2(x, \eta) \Delta_\kappa^\beta q(x)) \leq \epsilon_1 \|g_1\| + \epsilon_2 \|g_2\|,$$

which contradicts assumption (B). Hence, η is an efficient solution which minimizes problem (19). This completes the proof. \square

Theorem 3.2 *Suppose that the following hypotheses are achieved:*

- (A) η is a weakly efficient point of the solution $Q(t)$ of the optimal problem (19);
- (B) A couple $(\beta, \kappa) - (Q, q)$ is conformable univex of order $\beta \in [0, 1]$ in the direction of $\eta \in \mathfrak{J}$. Moreover, for some $\bar{x} \in \mathfrak{J}$ with $q(\bar{x}) < 0$, there occur two constants $\epsilon_1 \geq 0$ and $\epsilon_2 \geq 0$ satisfying

$$\epsilon_1 (f_1(x, \eta) \Delta_\kappa^\beta Q(x)) + \epsilon_2 (f_2(x, \eta) \Delta_\kappa^\beta q(x)) \geq 0.$$

Proof We aim to show that the system

$$f_1(x, \eta) \Delta_\kappa^\beta Q(x) < 0, \quad f_2(x, \eta) \Delta_\kappa^\beta q(x) < 0$$

has no solution for all $x \in \mathfrak{J}$. If the system has at least one solution $y \in \mathfrak{J}$, then in view of Definition 8, we get

$$Q(\eta + \rho_1 y) < Q(\eta) \quad \text{and} \quad q(\eta + \rho_2 y) < q(\eta)$$

for sufficient small arbitrary constants $\rho_1, \rho_2 > 0$. By letting $\bar{x} := \eta + \rho_2 y$, which leads to $\bar{x} \in \mathfrak{J} \cap N_{\rho_2}(\eta)$, in view of assumption (B), one can get

$$q(\eta + \rho_2 y) = q(\bar{x}) < 0,$$

which is a contradiction with assumption (A), where η is a weak solution ($q(\eta) \geq 0$). Hence, the above inequalities are nonnegative. Consequently, there exist two nonnegative con-

stants ϵ_1 and ϵ_2 achieving the conclusion

$$\epsilon_1(f_1(x, \eta)\Delta_\kappa^\beta Q(x)) + \epsilon_2(f_2(x, \eta)\Delta_\kappa^\beta q(x)) \geq 0,$$

with the property $\epsilon_2 q(\eta) \geq 0$.

This completes the proof. \square

Remark 2

- In Theorem 3.2, \bar{x} is called a feasible point, which appears in the solution space. This space represents the initial set of candidate solutions to the optimal problem;
- The optimal problem (19) is a generalization of recent works such as [21–23];
- Minimizing problem (19) is equivalent to minimizing the problem

$$\begin{aligned} &\text{Minimize } \mathcal{D}^\beta Q(t), \quad t \in J = [0, T] \\ &\text{subject to } Q_0 = q \leq 0; \end{aligned} \quad (22)$$

- Theorems 3.1 and 3.2 can be generalized to \mathbb{R}^n .

3.1 Numerical examples

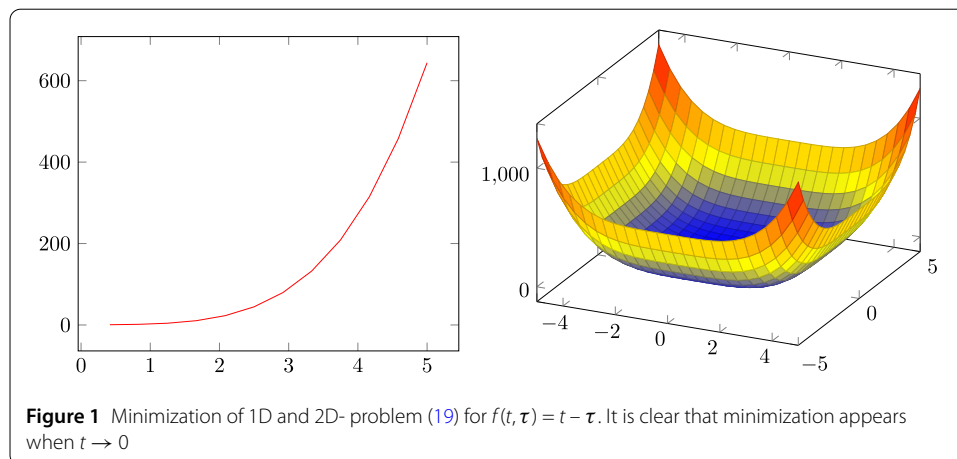
Consider the following data: $\Psi(v) = v$, $f(t, \tau) = t - \tau$, $g(t, \tau) = 1$, $\beta = \kappa = 0.5$ and $v_1(\beta, t) = (1 - \beta)t^\beta$, $v_0(\beta, t) = \beta t^{1-\beta}$, $t \in [0, 5]$. To minimize problem (19), we have the following calculation:

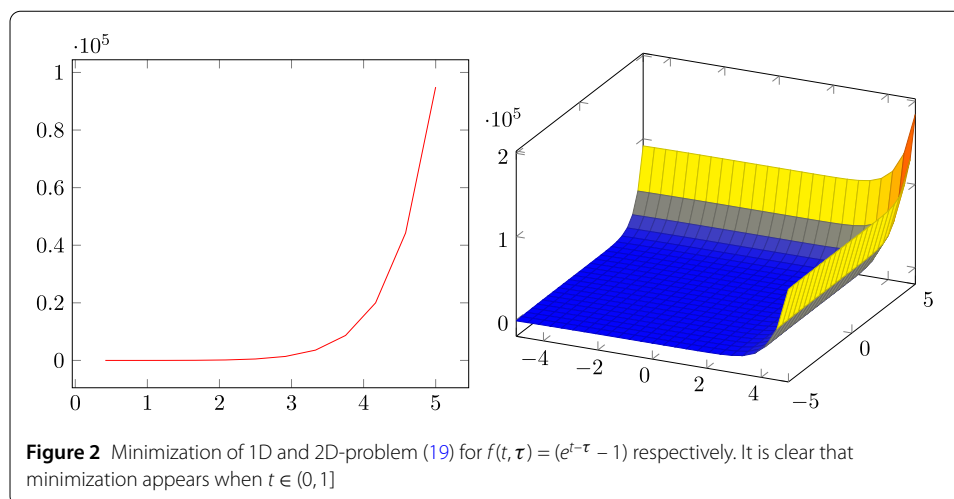
$$\Delta_\kappa^\beta Q(t) = \Delta_\kappa^\beta \sqrt{t^3} \approx t^4 + 0.7t^2 + 0.7\sqrt{t}, \quad t \in [0, 5],$$

where $A = B = C = 1$. It is clear that $Q(0) = 0$, and for arbitrary constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$, we get

$$f(t, \tau)\Delta_\kappa^\beta Q(t) = 5 * t^4 + 0.7 * t^2 + 0.7 * t^{0.5} > 1.$$

Thus, in view of Theorem 3.1, the minimization can be recognized when $t \rightarrow 0$ (see Fig. 1).





Suppose that $f(t, \tau) = (e^{t-\tau} - 1)$, $t - \tau \geq 1$. Then we obtain

$$f(t, \tau) \Delta_{\kappa}^{\beta} Q(t) = 1.5 * t^4 + 0.7 * t^2 + 0.7 * t^{0.5} > 1.$$

Hence, in view of Theorem 3.1, the minimization can be seen in Fig. 2. Similarly, one can optimize problem (19) by using Theorem 3.2.

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Competing interests

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Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details

¹Informetrics Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam. ²Faculty of Mathematics & Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam. ³Deanship of Educational Services, Department of General Sciences, Prince Sultan University, Riyadh, Saudi Arabia. ⁴School of Mathematical and Computer Sciences, Heriot-Watt University, Putrajaya, Malaysia.

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